Introductory Mathematics
Illustrative Problems and Solutions
(Part 2 of 3)

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11. Applied arithmetic and word problems

Applied arithmetic and word problems – example 1

Question

A two-digit number multiplied by the sum of its digits yields 1666. The tens digit is one greater than the units digit. What is the sum of the digits of the number?

Suppose the tens digit is x and the units digit is y. Note that $0 \leq x, y \leq 9$.

$$\left.\begin{array}{rcl}(10x+y)(x+y) & = & 1666 \\ x & = & y+1\end{array}\right\} \Longrightarrow x=9,\ y=8$$

Applied arithmetic and word problems – example 2

Question

The front wheel of a vehicle has a circumference of 2.1 meters, while the rear wheel has a circumference of 3.5 meters. If the rear wheel rotates 2000 fewer times than the front wheel over a certain distance, what is the total length of the track (in kilometers)?

Let the distance be x, then

$$\frac{x}{2.1} - \frac{x}{3.5} = 2000 \Longrightarrow x = 10500 \text{ m} = 10.5 \text{ km}$$

Applied arithmetic and word problems – example 3

Question

A tank can be filled by the first pump in 9 hours and by the second pump in 6 hours. A third pump is to be added such that all three pumps working together can fill the tank in 2 hours. How long would it take the third pump alone to fill the tank?

Let the answer be x, then

$$\frac{1}{9} + \frac{1}{6} + \frac{1}{x} = \frac{1}{2} \implies x = \frac{9}{2}$$

Applied arithmetic and word problems – example 4

Question

A fraction becomes $\frac{1}{3}$ when its numerator is decreased by 3 and its denominator is increased by 2. If the numerator is increased by 1 and the denominator is decreased by 1, the fraction becomes $\frac{3}{4}$. Which fraction satisfies both conditions?

Let the numerator be x and denominator be y

$$\left.\begin{aligned} \frac{x-3}{y+2} &= \frac{1}{3} \\ \frac{x+1}{y-1} &= \frac{3}{4} \end{aligned}\right\} \Longrightarrow x = 8,\ y = 13$$

Applied arithmetic and word problems – example 5

Question

A family has multiple children. One of the sons says, "I have as many brothers as I have sisters." One of the daughters says, "I have three times as many brothers as I have sisters." How many sons and daughters does the family have?

Suppose there are x sons and y daughters

$$\left.\begin{aligned} x-1 &= y \\ x &= 3(y-1) \end{aligned}\right\} \Longrightarrow x=3,\ y=2$$

Applied arithmetic and word problems – example 6

Question

Pump A can fill a tank in 12 minutes, while pump B can fill it in 24 minutes. Pump A is turned on first, and pump B is turned on 3 minutes later. From the moment they are turned on, both pumps operate continuously until the tank is completely filled. How long will the tank be filled?

Let x and y be the total operating time for A and B respectively

$$\left.\begin{array}{rcl} \dfrac{x}{12}+\dfrac{y}{24} & = & 1 \\ x & = & y+3 \end{array}\right\} \Longrightarrow x=9,\ y=6$$

Applied arithmetic and word problems – example 7

Question

A fraction becomes $\frac{1}{2}$ when its numerator is decreased by 2 and its denominator is increased by 1. If, instead, the numerator is increased by 2 and the denominator is decreased by 1, the fraction becomes $\frac{6}{7}$. Determine the fraction.

Let the numerator be x and denominator be y

$$\left.\begin{aligned} \frac{x-2}{y+1} &= \frac{1}{2} \\ \frac{x+2}{y-1} &= \frac{6}{7} \end{aligned}\right\} \Longrightarrow x = 10,\ y = 15$$

Applied arithmetic and word problems – example 8

Question

A gardener walked around a rectangular garden four times, covering a total distance of 160 meters. One side of the garden is 4 meters longer than the other. Determine the length and width of the garden.

Let the length and width of the garden be x and y respectively

$$\left.\begin{array}{rcl} x & = & y+4 \\ 4(2x+2y) & = & 160 \end{array}\right\} \Longrightarrow x = 12,\ y = 8$$

Applied arithmetic and word problems – example 9

Question

In a classroom, if students are seated such that each desk accommodates 7 students, the last desk ends up with only one student. However, if 6 students are seated at each desk, then one student remains without a seat. Determine the total number of students in the classroom.

Let x be the number of students and y be the number of desks.

$$\left.\begin{aligned} x &= 7(y-1)+1 \\ x &= 6y+1 \end{aligned}\right\} \Longrightarrow x = 43,\ y = 7$$

Applied arithmetic and word problems – example 10

Question

A school fundraiser collected a total of 130 euros to purchase 55 tickets to a cinema and a theatre. Cinema tickets cost 1 euro each, while theatre tickets cost 3.50 euros each. The treasurer forgot the exact number of each type of ticket to purchase. How many cinema tickets should the treasurer buy?

Let x be the number of cinema tickets, and y be the number of theatre tickets

$$\left.\begin{aligned} x + y &= 55 \\ x + 3.5y &= 130 \end{aligned}\right\} \implies \quad x = 25, \quad y = 30$$

Applied arithmetic and word problems – example 11

Question

A rectangle has a length that is 20cm longer than its width. If the length is decreased by 5cm and the width is increased by 10cm, the area of the rectangle increases by 300cm^2. What is the width of the rectangle?

Let x be the width in cm. Then the length is $x + 20$.

$$x(x + 20) + 300 = (x + 10)(x + 15) \implies x = 30$$

Applied arithmetic and word problems – example 12

Question

A gardener bought several trees for a total of 720 euros. If each tree had cost 2 euros less, the gardener could have bought 5 more trees for the same total price. How many trees did the gardener buy?

Let x be the number of trees the gardener originally bought. The cost per tree is $\frac{720}{x}$ euros. Then

$$\frac{720}{x+5} = \frac{720}{x} - 2 \Longrightarrow x = 40$$

Applied arithmetic and word problems – example 13

Question

At a youth gathering, the number of boys was three times the number of girls. After 8 boys and 8 girls left the meeting early, the number of remaining boys became five times the number of remaining girls. Determine the number of boys who originally attended the meeting.

Let b be the number of boys and g the number of girls.

$$\left.\begin{aligned} b &= 3g \\ b-8 &= 5(g-8) \end{aligned}\right\} \implies g = 16, \quad b = 48$$

Applied arithmetic and word problems – example 14

Question

A water tank can be filled by Pump A in 7.5 hours and by Pump B in 5 hours. If both pumps operate simultaneously, how long will it take to fill the entire tank?

Let the filling rates of the two pumps be:

$$\frac{1}{7.5} = \frac{2}{15}, \quad \frac{1}{5}$$

The combined rate is:

$$\frac{2}{15} + \frac{1}{5} = \frac{2}{15} + \frac{3}{15} = \frac{5}{15} = \frac{1}{3}$$

So, the time to fill the tank together is:

$$\frac{1}{\frac{1}{3}} = 3 \text{ hours}$$

Applied arithmetic and word problems – example 15

Question

A water tank can be filled by Pump A in 9 hours, by Pump B in 6 hours, and by Pump C in 4 hours. If all three pumps are operated simultaneously, how long will it take to fill the entire tank?

Let T be the time in hours to fill the tank together.

$$\frac{1}{9}+\frac{1}{6}+\frac{1}{4}=\frac{4+6+9}{36}=\frac{19}{36}\implies\frac{1}{T}=\frac{19}{36}\implies T=\frac{36}{19}\approx 1.89 \text{ hours}$$

Applied arithmetic and word problems – example 16

Question

A florist prepared a bouquet. The customer selected roses priced at 3.5 euros each and gerberas priced at 2 euros each. The bouquet contained 23 flowers in total and cost 58 euros. How many roses and gerberas were in the bouquet?

Let r be the number of roses and g the number of gerberas.

$$\left.\begin{array}{rcl} r+g & = & 23 \\ 3.5r+2g & = & 58 \end{array}\right\} \implies \quad r=8, \quad g=15$$

Applied arithmetic and word problems – example 17

Question

A water tank can be filled by three pumps. Pump A fills the tank in 9 hours, pump B in 6 hours, and pump C in 18 hours. How long will it take to fill the tank if all three pumps are operated together?

Let the filling rates of the three pumps be::

$$\frac{1}{9}, \ \frac{1}{6}, \ \frac{1}{18}$$

Then the time t satisfies

$$\frac{1}{t} = \frac{1}{9} + \frac{1}{6} + \frac{1}{18} = \frac{2}{18} + \frac{3}{18} + \frac{1}{18} = \frac{6}{18} = \frac{1}{3} \quad \Longrightarrow \quad t = 3$$

Applied arithmetic and word problems – example 18

Question

A florist assembled a bouquet consisting of roses and gerberas. Each rose cost 3 euros, and each gerbera cost 2 euros. The bouquet contained a total of 15 flowers. The total cost of the bouquet was 6 euros more than twice the number of flowers. How many roses and gerberas were in the bouquet?

Let r be the number of roses, and g the number of gerberas.

$$\left.\begin{array}{rcl} r+g & = & 15 \\ 3r+2g & = & 2\times 15+6=36 \end{array}\right\} \Longrightarrow g=9,\ r=6$$

Applied arithmetic and word problems – example 19

Question

A rectangular plot is to be enclosed on three sides using 20 meters of fencing. One of the longer sides is to remain unfenced. The area of the plot is 10 square meters less than twice the full perimeter of the rectangle. What is the length of the shorter side of the rectangle?

Let x be the length of the shorter side (width), and y the longer side (length).

$$\left.\begin{array}{rcl} 2x + y & = & 20 \\ xy & = & 4(x+y) - 10 \end{array}\right\} \Longrightarrow x = 5,\; y = 10 \text{ or } x = 7,\; y = 6$$

Since $x < y$, $x = 5$

Applied arithmetic and word problems – example 20

Question

Two workers, Worker A and Worker B, can complete a job together in 18 days. After working together for 15 days, Worker A falls ill, and Worker B completes the remaining work alone in 7.5 days. How many days would it take Worker B to complete the entire job alone?

Let x and y be the working rates for Worker A and Worker B respectively.

$$\left.\begin{array}{rcl} x+y & = & \frac{1}{18} \\ 15(x+y)+7.5y & = & 1 \end{array}\right\} \Longrightarrow x=\frac{1}{30},\ y=\frac{1}{45}$$

Answer: 45 days.

Applied arithmetic and word problems – example 21

Question

A swimming pool with a capacity of 990 hectoliters is being filled using two water taps. The first tap runs continuously for 8 hours, and the second tap runs simultaneously for 6 hours. The first tap has a flow rate that is 10 hectoliters per hour greater than that of the second tap. What is the flow rate of the first tap in hectoliters per hour?

Let x and y be the flow rates for the first tap and the second tap respectively.

$$\left.\begin{array}{rcl} 8x + 6y & = & 990 \\ x & = & y + 10 \end{array}\right\} \Longrightarrow x = 75, \; y = 65$$

Applied arithmetic and word problems – example 22

Question

Express the number 1086 as a sum of three addends such that the first is 267 greater than the second, and the third equals the sum of the first two. What is the value of the first addend?

Let the second addend be x. Then the first addend is $x + 267$ and the third addend is $x + (x + 267) = 2x + 267$. Total sum of addends is:

$$x + (x + 267) + (2x + 267) = 1086 \implies x = 138$$

The first addend is then $x + 267 = 405$ and the third $405 + 138 = 543$.

Applied arithmetic and word problems – example 23

Question

Express the number 1872 as a sum of three addends such that the second addend is three times the first, and the third equals the sum of the first two. What is the product of the digits of the first addend?

Let the first addend be x. Then the second addend is $3x$ and the third $x + 3x = 4x$. We have

$$x + 3x + 4x = 1872 \Longrightarrow x == 234$$

Product of digits of 234:

$$2 \times 3 \times 4 = 24$$

Applied arithmetic and word problems – example 24

Question

Express the number 1749 as a sum of three addends such that the second addend is three times the first, and the third addend is 123 less than the sum of the first two. What is the product of the digits of the first addend?

Let the first addend be x. Then the second addend is $3x$ and third $x + 3x - 123 = 4x - 123$.

$$x + 3x + (4x - 123) = 1749 \implies x = 234$$

Product of the digits of 234:

$$234 = 2 \cdot 3 \cdot 4 = 24.$$

Applied arithmetic and word problems – example 25

Question

Express the number 1968 as a sum of three addends such that the second addend is twice the first, and the third addend is 84 more than the first. What is the sum of the digits of the second addend?

Let the first addend be x. Then the second addend is $2x$ and the third $x + 84$

$$x + 2x + (x + 84) = 1968 \implies x = 471$$

The second addend is then

$$2 \times 471 = 942$$

Sum of the digits of 942

$$9 + 4 + 2 = 15$$

12. System of equations (linear and quadratic)

System of equations (linear and quadratic) – example 1

Question

Solve

$$
\begin{aligned}
x^2 - y^2 - 2 &= 0 \\
-x + y + 1 &= 0
\end{aligned}
$$

From the second equation we have

$$y = x - 1$$

substituting to the first one

$$x^2 - (x-1)^2 - 2 = 0 \Longrightarrow 2x - 3 = 0 \Longrightarrow x = \frac{3}{2}$$

Then

$$y = \frac{1}{2}$$

System of equations (linear and quadratic) – example 2

Question

Solve

$$\begin{aligned} x^2 - y^2 + 2 &= 0 \\ -x + y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = x - 1$$

substituting to the first one

$$x^2 - (x-1)^2 + 2 = 0 \Longrightarrow 2x + 1 = 0 \Longrightarrow x = -\frac{1}{2}$$

Then

$$y = -\frac{3}{2}$$

System of equations (linear and quadratic) – example 3

Question

Solve

$$\begin{aligned} x^2 - 2y^2 + 2 &= 0 \\ x - y - 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = x - 1$$

substituting to the first one

$$x^2 - 2(x-1)^2 + 2 = 0 \Longrightarrow -x^2 + 4x = 0 \Longrightarrow x = 0,\ 4$$

Then

$$y = -1,\ 3$$

System of equations (linear and quadratic) – example 4

Question

Solve

$$\begin{aligned} 2x^2 - y^2 + 2 &= 0 \\ x - y - 3 &= 0 \end{aligned}$$

From the second equation we have

$$y = x - 3$$

substituting to the first one

$$2x^2 - (x-3)^2 + 2 = 0 \Longrightarrow (x+3)^2 = 16 \Longrightarrow x = -7,\ 1$$

Then

$$y = -10,\ -2$$

System of equations (linear and quadratic) – example 5

Question

Solve

$$\begin{aligned} 2x^2 - y^2 + 2 &= 0 \\ x - y + 3 &= 0 \end{aligned}$$

From the second equation we have

$$y = x + 3$$

substituting to the first one

$$2x^2 - (x+3)^2 + 2 = 0 \Longrightarrow x^2 - 6x - 7 = 0 \Longrightarrow x = 7, \ -1$$

Then

$$y = -10, \ 2$$

System of equations (linear and quadratic) – example 6

Question

Solve

$$\begin{aligned} 2x^2 - y^2 + 2 &= 0 \\ 3x + 2y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = -\frac{3x+1}{2}$$

substituting to the first one

$$2x^2 - \left(\frac{3x+1}{2}\right)^2 + 2 = 0 \Longrightarrow (x+3)^2 = 16 \Longrightarrow x = -7,\ 1$$

Then

$$y = 10,\ -2$$

System of equations (linear and quadratic) – example 7

Question

Solve

$$\begin{aligned} 2x^2 - y^2 - 2 &= 0 \\ 3x + 2y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = -\frac{3x+1}{2}$$

substituting to the first one

$$2x^2 - \left(\frac{3x+1}{2}\right)^2 - 2 = 0 \Longrightarrow x = -3$$

Then

$$y = -\frac{-9+1}{2} = 4$$

System of equations (linear and quadratic) – example 8

Question

Solve

$$\begin{aligned} x^2 + 2y^2 - 2 &= 0 \\ 2x + y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = -2x - 1$$

substituting to the first one

$$x^2 + 2(2x+1)^2 - 2 = 0 \Longrightarrow x = 0, \; -\frac{8}{9}$$

Then

$$y = -1, \; \frac{7}{9}$$

System of equations (linear and quadratic) – example 9

Question

Solve

$$\begin{aligned} x^2 - 3y^2 - 1 &= 0 \\ x + 3y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = -\frac{x+1}{3}$$

substituting to the first one

$$x^2 - 3\left(-\frac{x+1}{3}\right)^2 - 1 = 0 \Longrightarrow x = -1,\ 2$$

Then

$$y = 0,\ -1$$

System of equations (linear and quadratic) – example 10

Question

Solve

$$\begin{aligned} x^2 - y^2 - 1 &= 0 \\ x + 3y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = -\frac{x+1}{3}$$

substituting to the first one

$$x^2 - \left(-\frac{x+1}{3}\right)^2 - 1 = 0 \Longrightarrow x = -1,\ \frac{5}{4}$$

Then

$$y = 0,\ -\frac{3}{4}$$

System of equations (linear and quadratic) – example 11

Question

Solve

$$\begin{aligned} x^2 - y^2 - 1 &= 0 \\ x + 2y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = -\frac{x+1}{2}$$

substituting to the first one

$$x^2 - \left(-\frac{x+1}{2}\right)^2 - 1 = 0 \Longrightarrow x = -1,\ \frac{5}{3}$$

Then

$$y = 0,\ -\frac{4}{3}$$

System of equations (linear and quadratic) – example 12

Question

Solve

$$\begin{aligned} x^2 - 2y^2 - 1 &= 0 \\ x + 2y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = -\frac{x+1}{2}$$

substituting to the first one

$$x^2 - 2\left(-\frac{x+1}{2}\right)^2 - 1 = 0 \Longrightarrow x = -1,\ 3$$

Then

$$y = 0,\ -2$$

System of equations (linear and quadratic) – example 13

Question

Solve

$$\begin{aligned} x^2 - 4y^2 - 1 &= 0 \\ x + 2y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = -\frac{x+1}{2}$$

substituting to the first one

$$x^2 - 4\left(-\frac{x+1}{2}\right)^2 - 1 = 0 \Longrightarrow x = -1$$

Then

$$y = 0$$

System of equations (linear and quadratic) – example 14

Question

Solve

$$\begin{aligned} x^2 - 5y^2 - 1 &= 0 \\ x + 3y + 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = -\frac{x+1}{3}$$

substituting to the first one

$$x^2 - 5\left(-\frac{x+1}{3}\right)^2 - 1 = 0 \Longrightarrow x = -1,\ \frac{7}{2}$$

Then

$$y = 0,\ -\frac{3}{2}$$

System of equations (linear and quadratic) – example 15

Question

Solve

$$\begin{aligned} x^2 - 5y^2 - 1 &= 0 \\ x - 3y - 1 &= 0 \end{aligned}$$

From the second equation we have

$$y = \frac{x-1}{3}$$

substituting to the first one

$$x^2 - 5\left(\frac{x-1}{3}\right)^2 - 1 = 0 \Longrightarrow x = -\frac{7}{2},\ 1$$

Then

$$y = -\frac{3}{2},\ 0$$

13. Domains and ranges of radical, logarithmic, and rational functions

Domains and ranges of functions– example 1

Question

Let

$$f(x) = \frac{3-5x}{x^2-5x+7}.$$

Determine the set of all values of $x \in \mathbb{R}$ for which $f(x) \geq 0$.

Note that

$$x^2 - 5x + 7 = \left(x - \frac{5}{2}\right)^2 + \frac{3}{4} > 0$$

Thus, we need

$$3 - 5x \geq 0 \Longrightarrow x \leq \frac{3}{5}$$

Domains and ranges of functions – example 2

Question

Let

$$f(x) = \frac{3}{\sqrt{3^x + 1}}.$$

Determine the range of the function f.

Since $3^x > 0$ for all $x \in \mathbb{R}$, it follows that

$$\sqrt{3^x + 1} > \sqrt{1} = 1.$$

Therefore, the denominator of the function is always greater than 1, so

$$f(x) = \frac{3}{\sqrt{3^x + 1}} < 3.$$

Also, since both the numerator and the denominator are positive for all $x \in \mathbb{R}$, we conclude that

$$f(x) > 0.$$

Thus, $f(x) \in (0, 3)$.

Domains and ranges of functions – example 3

Question

Determine the domain of the function

$$f(x) = \sqrt{x + x^2 - 2x^3}.$$

$$\begin{aligned} x + x^2 - 2x^3 &\geq 0 \\ x(-2x^2 + x + 1) &\geq 0 \\ x\left(x + \frac{1}{2}\right)(x - 1) &\geq 0 \end{aligned}$$

$$\implies 0 \leq x \leq 1 \text{ or } x \leq -\frac{1}{2}$$

Domains and ranges of functions – example 4

Question

Determine the domain of the function

$$f(x) = \sqrt{\frac{7-x}{x-2}} + 2.$$

$$\frac{7-x}{x-2} \geq 0 \Longrightarrow 2 < x \leq 7$$

Domains and ranges of functions – example 5

Question

Determine the domain of the function

$$f(x) = \frac{1}{\sqrt{x - |x|}}.$$

Since

$$x - |x| \leq 0,$$

the domain is $\emptyset$.

Domains and ranges of functions – example 6

Question

Determine the domain of the function

$$f(x) = \log_3(x-2) + \sqrt{4x - x^2}.$$

$$\left.\begin{array}{rcl} x-2 & > & 0 \\ x(4-x) & \geq & 0 \end{array}\right\} \Longrightarrow 2 < x \leq 4$$

Domains and ranges of functions – example 7

Question

Determine the domain of the function

$$f(x) = \frac{\log(3 - 2x - x^2)}{\sqrt{x}}.$$

$$3 - 2x - x^2 = (x+3)(1-x)$$

$$\left.\begin{array}{rcl} (x+3)(1-x) & > & 0 \\ x & > & 0 \end{array}\right\} \Longrightarrow 0 < x < 1$$

Domains and ranges of functions – example 8

Question

Determine the domain of the function

$$f(x) = \sqrt{\frac{\log(5-x)}{9-2x}}.$$

$$\log(5-x) \geq 0 \iff 5-x \geq 1 \iff x \leq 4$$

The domain is given by

$$x \leq 4 \text{ and } 9 > 2x$$

or

$$4 < x < 5 \text{ and } 9 < 2x$$

i.e.

$$x \leq 4 \text{ or } \frac{9}{2} < x < 5$$

Domains and ranges of functions – example 9

Question

Determine the domain of the function

$$f(x) = \log\left(1 - \sqrt{4 - x^2}\right).$$

$$4 - x^2 \geq 0 \implies -2 \leq x \leq 2$$

$$1 - \sqrt{4 - x^2} > 0 \implies 4 - x^2 < 1 \implies x < -\sqrt{3} \text{ or } x > \sqrt{3}$$

Thus the domain is given by

$$-2 \leq x < -\sqrt{3} \text{ or } \sqrt{3} < x \leq 2$$

Domains and ranges of functions – example 10

Question

Determine the domain of the function

$$f(x) = \frac{3}{\sqrt{2^x - 1}}.$$

$$2^x - 1 > 0 \Longrightarrow x > 0$$

Domains and ranges of functions – example 11

Question

Determine the domain of the function

$$f(x) = \frac{2}{\sqrt{x^2 - 1}}.$$

$$x^2 - 1 > 0 \implies x < -1 \text{ or } x > 1$$

Domains and ranges of functions – example 12

Question

Determine the domain of the function

$$f(x) = \log\left(1 - \sqrt{5 - x^2}\right).$$

$$0 \leq 5 - x^2 < 1 \implies -\sqrt{5} \leq x < -2 \text{ or } 2 < x \leq \sqrt{5}$$

Domains and ranges of functions – example 13

Question

Determine the domain of the function

$$f(x) = \frac{2}{\sqrt{1-x^2}}.$$

$$1 - x^2 > 0 \Longrightarrow -1 < x < 1$$

Domains and ranges of functions – example 14

Question

Determine the domain of the function

$$f(x) = \ln(x-1) + \sqrt{4x - x^2}.$$

$$x - 1 > 0 \implies x > 1$$

$$4x - x^2 \geq 0 \implies 0 \leq x \leq 4$$

Thus

$$1 < x \leq 4$$

Domains and ranges of functions – example 15

Question

Determine the domain of the function

$$f(x) = \frac{\log(-3 - 2x + x^2)}{\sqrt{x}}.$$

$$-3 - 2x + x^2 > 0 \Longrightarrow x < -1 \text{ or } x > 3$$

Together with $x > 0$, we get

$$x > 3$$

Domains and ranges of functions – example 16

Question

Determine the domain of the function

$$f(x) = \frac{\log(-3 - 2x + x^2)}{\sqrt{1 - x}}.$$

$$-3 - 2x + x^2 > 0 \implies x < -1 \text{ or } x > 3$$

$$1 - x > 0 \implies x < 1$$

In conclusion

$$x < -1$$

Domains and ranges of functions – example 17

Question

Determine the domain of the function

$$f(x) = \frac{1}{\sqrt{|x| - x}}.$$

$$|x| - x > 0 \Longrightarrow x < 0$$

14. Function behavior and properties

Function behavior and properties

Question

For each of the given functions f, answer

- Is the function even or odd?
- Is the function bounded above and/or below?
- Is the function increasing or decreasing? On which intervals?
- Is the function injective (one-to-one)?
- Is the function surjective (onto)?
- Is the function periodic?

Function behavior and properties – example 1

$$f(x) = |3 - x| + 2.$$

$$|3 - x| + 2 = \begin{cases} 5 - x & \text{if } x < 3 \\ x - 1 & \text{if } x \geq 3 \end{cases}$$

- Strictly decreasing on the interval $(-\infty, 3]$
- Strictly increasing on the interval $[3, \infty)$
- Attains a global minimum at $x = 3$, where $f(3) = 2$
- Bounded below by 2; not bounded above
- Neither even nor odd
- Not injective on $\mathbb{R}$, but injective on $(-\infty, 3]$ and on $[3, \infty)$
- Surjective onto $[2, \infty)$
- Not periodic

Function behavior and properties – example 2

$$f(x) = 2 - \frac{1}{x}.$$

- Neither even nor odd
- Not bounded above or below
- Strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$
- Not defined at $x = 0$
- Injective on $\mathbb{R} \setminus \{0\}$
- Surjective onto its range $\mathbb{R} \setminus \{2\}$
- Not periodic

Function behavior and properties – example 3

$$f(x) = \frac{1}{x+2}.$$

- Neither even nor odd
- Not bounded above or below
- Strictly decreasing on $(-\infty, -2)$ and on $(-2, \infty)$
- Not defined at $x = -2$
- Injective on $\mathbb{R} \setminus \{-2\}$
- Surjective onto $\mathbb{R} \setminus \{0\}$
- Not periodic

Function behavior and properties – example 4

$$f(x) = \cot\left(x + \frac{\pi}{2}\right).$$

- Odd function
- Not bounded above or below
- Strictly decreasing on each interval $\left(k\pi - \frac{\pi}{2}, (k+1)\pi - \frac{\pi}{2}\right)$, where $k \in \mathbb{Z}$
- Not defined at $x = -\frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$
- Bijective on each interval $\left(k\pi - \frac{\pi}{2}, (k+1)\pi - \frac{\pi}{2}\right)$, where $k \in \mathbb{Z}$
- Periodic

Function behavior and properties – example 5

$$f(x) = -\log_{\frac{1}{2}}(x).$$

- Neither even nor odd
- Not bounded above or below
- Strictly increasing on its domain $(0, \infty)$
- Injective on $(0, \infty)$
- Surjective onto $\mathbb{R}$
- Not periodic

Function behavior and properties – example 6

$$f(x) = \sqrt{1 - x}$$

- Neither even nor odd
- Bounded below by 0; not bounded above
- Strictly decreasing on its domain $(-\infty, 1]$
- Injective on $(-\infty, 1]$
- Surjective onto $[0, \infty)$
- Not periodic

Function behavior and properties – example 7

$$f(x) = 2^{1-x}$$

- Neither even nor odd
- Bounded below by 0; not bounded above
- Strictly decreasing on $\mathbb{R}$
- Injective on $\mathbb{R}$
- Surjective onto $(0, \infty)$
- Not periodic

Function behavior and properties – example 8

$$f(x) = x^4$$

- Even function
- Bounded below by 0; not bounded above
- Strictly decreasing on $(-\infty, 0]$; strictly increasing on $[0, \infty)$
- Not injective on $\mathbb{R}$, but injective on $(-\infty, 0]$ and on $[0, \infty)$
- Surjective onto $[0, \infty)$
- Not periodic

Function behavior and properties – example 9

$$f(x) = x^3$$

- Odd function
- Not bounded
- Strictly increasing on $\mathbb{R}$
- Injective on $\mathbb{R}$
- Surjective onto $\mathbb{R}$
- Not periodic

Function behavior and properties – example 10

$$f(x) = -x^3$$

- Odd function
- Not bounded
- Strictly decreasing on $\mathbb{R}$
- Injective on $\mathbb{R}$
- Surjective onto $\mathbb{R}$
- Not periodic

Function behavior and properties – example 11

$$f(x) = \cot(x)$$

- Odd function
- Not bounded
- Strictly decreasing on each open interval $(k\pi, (k+1)\pi)$, where $k \in \mathbb{Z}$
- Bijective on each interval $(k\pi, (k+1)\pi)$, where $k \in \mathbb{Z}$
- Not defined at $x = k\pi,\ k \in \mathbb{Z}$
- Periodic

Function behavior and properties – example 12

$$f(x) = \sin(x)$$

- Odd function
- Bounded between -1 and 1
- Increasing on intervals $\left[-\frac{\pi}{2} + 2k\pi,\ \frac{\pi}{2} + 2k\pi\right]$, decreasing on intervals $\left[\frac{\pi}{2} + 2k\pi,\ \frac{3\pi}{2} + 2k\pi\right]$, where $k \in \mathbb{Z}$
- Not injective on $\mathbb{R}$, but injective on each interval $\left[\frac{\pi}{2} + 2k\pi,\ \frac{3\pi}{2} + 2k\pi\right]$, where $k \in \mathbb{Z}$
- Surjective onto $[-1, 1]$
- Periodic

Function behavior and properties – example 13

$$f(x) = \cos(x)$$

- Even function
- Bounded between -1 and 1
- Strictly decreasing on each interval $[2k\pi, (2k+1)\pi]$ and strictly increasing on $[(2k+1)\pi, (2k+2)\pi]$, where $k \in \mathbb{Z}$
- Not injective on $\mathbb{R}$, but injective on each interval $[2k\pi, (2k+1)\pi]$ and on $[(2k+1)\pi, (2k+2)\pi]$, where $k \in \mathbb{Z}$
- Surjective onto $[-1, 1]$
- Periodic

Function behavior and properties – example 14

$$f(x) = x^2 - 5$$

- Even function
- Bounded below by -5, not bounded above
- Strictly decreasing on $(-\infty, 0]$, strictly increasing on $[0, \infty)$
- Not injective on $\mathbb{R}$, but injective on $(-\infty, 0]$ and on $[0, \infty)$
- Surjective onto $[-5, \infty)$
- Not periodic

Function behavior and properties – example 15

$$f(x) = 10^x$$

- Neither even nor odd
- Bounded below by 0, not bounded above
- Strictly increasing on $\mathbb{R}$
- Injective on $\mathbb{R}$
- Surjective onto $(0, \infty)$
- Not periodic

Function behavior and properties – example 16

$$f(x) = |x - 2|$$

- Neither even nor odd
- Bounded below by 0, not bounded above
- Strictly decreasing on $(-\infty, 2]$, strictly increasing on $[2, \infty)$
- Not injective on $\mathbb{R}$, but injective on $(-\infty, 2]$ and on $[2, \infty)$
- Surjective onto $[0, \infty)$
- Not periodic

Function behavior and properties – example 17

$$f(x) = \tan(2x)$$

- Odd function
- Not bounded
- Strictly increasing on each open interval $\left(\frac{(2k-1)\pi}{4}, \frac{(2k+1)\pi}{4}\right)$, where $k \in \mathbb{Z}$
- Bijective on each such interval
- Periodic

Function behavior and properties – example 18

$$f(x) = x$$

- Odd function
- Not bounded
- Strictly increasing on $\mathbb{R}$
- Bijective on $\mathbb{R}$
- Not periodic

Function behavior and properties – example 19

$$f(x) = 1 - x$$

- Neither even nor odd
- Not bounded
- Strictly decreasing on $\mathbb{R}$
- Bijective on $\mathbb{R}$
- Not periodic

Function behavior and properties – example 20

$$f(x) = |x|$$

- Even function
- Bounded below by 0; not bounded above
- Strictly decreasing on $(-\infty, 0]$, strictly increasing on $[0, \infty)$
- Not injective on $\mathbb{R}$, but injective on each of the intervals $(-\infty, 0]$ and $[0, \infty)$
- Surjective onto $[0, \infty)$
- Not periodic

Function behavior and properties – example 21

$$f(x) = 4^x$$

- Neither even nor odd
- Strictly increasing on $\mathbb{R}$
- Bounded below by 0, not bounded above
- Injective on $\mathbb{R}$
- Surjective onto $(0, \infty)$
- Not periodic

Function behavior and properties – example 22

$$f(x) = -3^x$$

- Neither even nor odd
- Strictly decreasing on $\mathbb{R}$
- Bounded above by 0, not bounded below
- Injective on $\mathbb{R}$
- Surjective onto $(-\infty, 0)$
- Not periodic

Function behavior and properties – example 23

$$f(x) = 1 + \frac{1}{x-1}$$

- Neither even nor odd
- Not bounded above or below
- Strictly decreasing on both intervals $(-\infty, 1)$ and $(1, \infty)$
- Not defined at $x = 1$
- Injective on its domain $\mathbb{R} \setminus \{1\}$
- Surjective onto $\mathbb{R}$
- Not periodic

Function behavior and properties – example 24

$$f(x) = -\frac{1}{x+2}$$

- Neither even nor odd
- Not bounded above or below
- Strictly increasing on both intervals $(-\infty, -2)$ and $(-2, \infty)$
- Not defined at $x = -2$
- Injective on its domain $\mathbb{R} \setminus \{-2\}$
- Surjective onto $\mathbb{R}$
- Not periodic

Function behavior and properties – example 25

$$f(x) = \cos\left(x + \frac{\pi}{2}\right)$$

- Even function
- Bounded between -1 and 1
- Strictly decreasing on intervals $[2k\pi - \frac{\pi}{2}, (2k+1)\pi - \frac{\pi}{2}]$ and $[(2k+1)\pi - \frac{\pi}{2}, (2k+2)\pi - \frac{\pi}{2}]$, where $k \in \mathbb{Z}$
- Injective on each such interval $[2k\pi - \frac{\pi}{2}, (2k+1)\pi - \frac{\pi}{2}]$ and $[(2k+1)\pi - \frac{\pi}{2}, (2k+2)\pi - \frac{\pi}{2}]$, where $k \in \mathbb{Z}$
- Surjective onto $[-1, 1]$
- Periodic

Function behavior and properties – example 26

$$f(x) = -\log_2 x$$

- Neither even nor odd
- Not bounded
- Strictly decreasing on $(0, \infty)$
- Injective on its domain $(0, \infty)$
- Surjective onto $\mathbb{R}$
- Not periodic

Function behavior and properties – example 27

$$f(x) = -\sqrt{1-x}$$

- Neither even nor odd
- Bounded above by 0
- Strictly increasing on $(-\infty, 1]$
- Injective on its domain $(-\infty, 1]$
- Surjective onto $(-\infty, 0]$
- Not periodic

Function behavior and properties – example 28

$$f(x) = 2^{1+x}$$

- Neither even nor odd
- Bounded below by 0, not bounded above
- Strictly increasing on $\mathbb{R}$
- Injective on $\mathbb{R}$
- Surjective onto $(0, \infty)$
- Not periodic

Function behavior and properties – example 29

$$f(x) = -\frac{1}{x}$$

- Odd function
- Not bounded above or below
- Strictly decreasing on both intervals $(-\infty, 0)$ and $(0, \infty)$
- Not defined at $x = 0$
- Injective on its domain $\mathbb{R} \setminus \{0\}$
- Surjective onto $\mathbb{R}$
- Not periodic

Function behavior and properties – example 30

$$f(x) = \frac{1}{x-1}$$

- Neither even nor odd
- Not bounded above or below
- Strictly decreasing on both intervals $(-\infty, 1)$ and $(1, \infty)$
- Injective on its domain $\mathbb{R} \setminus \{1\}$
- Surjective onto $\mathbb{R}$
- Not periodic

Function behavior and properties – example 31

$$f(x) = x^2 - 5$$

- Even function
- Bounded below by -5, not bounded above
- Strictly decreasing on $(-\infty, 0]$, strictly increasing on $[0, \infty)$
- Not injective on $\mathbb{R}$, but injective on $(-\infty, 0]$ and $[0, \infty)$
- Surjective onto $[-5, \infty)$
- Not periodic

Function behavior and properties – example 31

$$f(x) = -\cot(2x)$$

- Odd function
- Not bounded above or below
- Strictly increasing on each open interval $\left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$, where $k \in \mathbb{Z}$
- Not defined at $x = \frac{k\pi}{2}$, where $k \in \mathbb{Z}$
- Injective on each interval $\left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$
- Surjective onto $\mathbb{R}$ over each such interval
- Periodic with period $\frac{\pi}{2}$

Function behavior and properties – example 32

$$f(x) = \log(x + 1)$$

- Neither even nor odd
- Not bounded above or below
- Strictly increasing on its domain $(-1, \infty)$
- Injective on its domain
- Surjective onto $\mathbb{R}$
- Not periodic

Function behavior and properties – example 33

$$f(x) = -3^{x+1}$$

- Neither even nor odd
- Strictly decreasing on $\mathbb{R}$
- Bounded above by -1, not bounded below
- Injective on $\mathbb{R}$
- Surjective onto $(-\infty, -1)$
- Not periodic

Function behavior and properties – example 34

$$f(x) = 1 - \frac{2}{x-3}$$

- Neither even nor odd
- Not bounded above or below
- Strictly increasing on each interval $(-\infty, 3)$ and $(3, \infty)$
- Not defined at $x = 3$
- Injective on its domain $\mathbb{R} \setminus \{3\}$
- Surjective onto $\mathbb{R}$
- Not periodic

15. Combinatorics

Combinatorics – example 1

Question

Five boys are to sit on a bench. Two of them are friends and must always sit next to each other. In how many distinct ways can all five boys be seated so that the two friends are always adjacent?

Treat the two friends as a single block. Then:

- There are $4!$ ways to arrange the block and the 3 other boys.
- The two friends can swap seats within their block: $2!$ ways.

$$\text{Total arrangements} = 4! \times 2! = 24 \times 2 = 48$$

Combinatorics – example 2

Question

There are 7 books to be arranged on a shelf, three of which are distinct dictionaries. In how many different ways can the books be arranged if the dictionaries must always be placed next to each other?

Treat the 3 dictionaries as one block. Then we have 4 other books $+1$ block $= 5$ items to arrange:

$$5! = 120$$

There are $3!$ ways to arrange the 3 dictionaries.

$$120 \times 3! = 720$$

Combinatorics – example 3

Question

Four boys and four girls go to the cinema. They want to sit in a single row such that no two boys and no two girls sit next to each other. In how many different ways can they be seated to satisfy this condition?

The only valid arrangement pattern is alternating:
Boy–Girl–Boy–Girl–Boy–Girl–Boy–Girl
or
Girl–Boy–Girl–Boy–Girl–Boy–Girl–Boy

In each pattern:
- Boys can be arranged among themselves in $4!$ ways
- Girls can be arranged among themselves in $4!$ ways

$$2 \times 4! \times 4! = 2 \times 24 \times 24 = 1152$$

Combinatorics – example 4

Question

In how many different ways can we arrange 4 identical detective novels, 2 identical romance novels, and 3 identical poetry collections on a shelf?

We are arranging a total of:

$$4 + 2 + 3 = 9 \text{ books}$$

with repetitions: - 4 identical detective novels - 2 identical romance novels - 3 identical poetry collections

$$\frac{9!}{4! \times 2! \times 3!} = \frac{5 \times 6 \times 7 \times 8 \times 9}{2 \times 6} = 5 \times 7 \times 4 \times 9 = 1260$$

Combinatorics – example 5

Question

In a group of 11 students, 6 are underperforming. In how many ways can we select 4 students such that at most one of them is underperforming?

Case 1: 0 underperforming Choose 4 from the 5 performing students:

$$\binom{5}{4} = 5$$

Case 2: 1 **underperforming** Choose 1 from the 6 underperforming and choose 3 from the 5 performing:

$$\binom{6}{1} \times \binom{5}{3} = 6 \times 10 = 60$$

Total $60 + 5 = 65$.

Combinatorics – example 6

Question

In a group of 12 students, 6 are underperforming. In how many ways can we select 5 students such that exactly two of them are underperforming?

Choose 2 underperforming

$$\binom{6}{2} = 15$$

Choose 3 well-performing

$$\binom{6}{3} = 20$$

Total valid combinations:

$$15 \times 20 = 300$$

Combinatorics – example 7

Question

In a group of 13 students, 6 are underperforming. In how many ways can we select 6 students such that at most one of them is underperforming?

Choose 0 underperforming:

$$\binom{7}{6} = 7$$

Choose 1 underperforming:

$$\binom{6}{1} \times \binom{7}{5} = 6 \times 21 = 126$$

Total valid combinations:

$$7 + 126 = 133$$

Combinatorics – example 8

Question

In a group of 10 students, 5 are underperforming. In how many ways can we select 5 students such that exactly four of them are underperforming?

Choose 4 underperforming:

$$\binom{5}{4} = 5$$

Choose 1 well-performing:

$$\binom{5}{1} = 5$$

Total:

$$5 \times 5 = 25$$

Combinatorics – example 9

Question

In a group of 12 students, 6 are underperforming. In how many ways can we select 6 students such that at most one of them is underperforming?

Choose 0 underperforming:

$$\binom{6}{6} = 1$$

Choose 1 underperforming:

$$\binom{6}{1} \times \binom{6}{5} = 6 \times 6 = 36$$

Total valid combinations:

$$1 + 36 = 37$$

Combinatorics – example 10

Question

Five boys want to sit on a bench. Two of them are upset with each other and refuse to sit next to each other. In how many ways can all 5 boys sit on the bench such that the two upset boys never sit together?

Total arrangements without restriction: $5! = 120$
Treat the two upset boys as a block (they sit together): Arrange the block and other 3 boys:

$$4! = 24$$

Internal swap within the block: $2! = 2$ Total with upset boys together:

$$24 \times 2 = 48$$

Valid arrangements where they are not together:

$$120 - 48 = 72$$

Combinatorics – example 11

Question

We have 6 tiles in three different colors, with two tiles of each color. In how many distinct ways can we arrange them in a row?

Total permutations of 6 tiles (with repetitions):

$$\frac{6!}{2! \times 2! \times 2!} = \frac{720}{8} = 90$$

Combinatorics – example 12

Question

We have 8 tiles in three colors: 2 white, 3 red, and 3 green. In how many distinct ways can we arrange them in a row?

$$\frac{8!}{2! \times 3! \times 3!} = \frac{4 \times 5 \times 6 \times 7 \times 8}{2 \times 6} = 2 \times 5 \times 7 \times 8 = 560$$

Combinatorics – example 13

Question

We have 8 tiles in three colors: 1 white, 3 red, and 4 green. In how many distinct ways can we arrange them in a row?

$$\frac{8!}{1! \times 3! \times 4!} = \frac{5 \times 6 \times 7 \times 8}{1 \times 6} = 5 \times 7 \times 8 = 280$$

Combinatorics – example 14

Question

Six girls want to sit on a bench. Two of them are upset with each other and refuse to sit next to each other. In how many distinct ways can the 6 girls sit on the bench so that those two never sit together?

Total unrestricted arrangements of 6 girls:

$$6! = 720$$

Treat the two upset girls as a block:
- Arrange the block $+$ remaining 4 girls: $5!$
- Swap the two upset girls within the block: $2!$

$$5! \times 2! = 120 \times 2 = 240$$

Valid arrangements (not sitting together):

$$720 - 240 = 480$$

Combinatorics – example 15

Question

Seven girls want to sit on a bench. Three of them are dressed identically and want to sit next to each other. In how many distinct ways can all 7 girls sit on the bench while satisfying this condition?

Treat the 3 identical-dress girls as a single block.
Arrange the 5 blocks:

$$5! = 120$$

Arrange the 3 girls within the block:

$$3! = 6$$

Total arrangements:

$$120 \times 6 = 720$$

Combinatorics – example 16

Question

At a concert, 7 girls want to sit in a single row. Four of them form a group and want to sit next to each other. In how many distinct ways can all 7 girls be seated while satisfying this condition?

Treat the group of 4 girls as a single block.
Arrange the 4 blocks:

$$4! = 24$$

Arrange the 4 girls within their group:

$$4! = 24$$

Total arrangements:

$$24 \times 24 = 576$$

Combinatorics – example 17

Question

Six girls want to sit on a bench. Among them, Jana, Maja, and Klára want to sit together as a group, but with the condition that Jana and Klára must always sit next to each other. In how many distinct ways can the 6 girls be seated while satisfying this condition?

Treat Jana and Klára as a fixed pair – 2 options (J–K or K–J).
Add Maja to form the group of 3 – 2 positions relative to the J–K pair
Total arrangements within the group:

$$2 \times 2 = 4$$

Treat the group as 1 block, plus 3 remaining girls – 4 blocks in total
Arrange the 4 blocks:

$$4! = 24$$

Total valid arrangements:

$$4 \times 24 = 96$$

Combinatorics – example 18

Question

At a concert, 3 married couples want to sit in a single row such that each husband and wife always sit next to each other. In how many distinct ways can they be seated while respecting this condition?

Treat each couple as a single block, in total 3 blocks Arrange the blocks:

$$3! = 6$$

Each couple can switch seats within their pair:

$$2 \times 2 \times 2 = 8$$

Total arrangements:

$$6 \times 8 = 48$$